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A Method of Evaluation of the Function $K_{is}(x)$

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1. Integral Representations of the Function $K_{is}(x)$

It is well known that the modified Bessel function of the second kind $K_\nu(x)$ can be expressed in the following forms [1]:

$$K_\nu(x) = \frac{1}{\cos\frac{\nu\pi}{2}} \int_0^\infty \cos(x \sinh t)\cosh(\nu t)\, dt, \tag{1.1}$$

and

$$K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh(\nu t)\, dt, \tag{1.2}$$

where $x > 0$ and $-1 < R(\nu) < 1$.

When $\nu = is$, where s is real, these two become as follows:

$$K_{is}(x) = \frac{1}{\cosh\frac{s\pi}{2}} \int_0^\infty \cos(x \sinh t) \cos(st)\, dt, \tag{1.3}$$

and

$$K_{is}(x) = \int_0^\infty e^{-x \cosh t} \cos(st)\, dt. \tag{1.4}$$

2. Numerical Computation of the Function $K_{is}(x)$

In order to compute the integral of (1.4) directly by the use of a quadrature formula of Gauss-Laguerre type, i.e.

$$\int_0^{\infty} e^{-t} f(t)\, dt = \sum_{i=1}^{n} a_i f(t_i), \tag{2.1}$$

it is necessary to write

$$K_{is}(x) = \int_0^{\infty} e^{-t} \left(e^{-x \cosh t + t} \cos(st)\right) dt, \tag{2.2}$$

or

$$K_{is}(x) = e^{-x} \int_0^{\infty} e^{-v} f(v)\, dv, \tag{2.3}$$

where

$$f(v) = \cos(s \log(\tfrac{1}{x} (v + x + \sqrt{v(v + 2x)}))/ \sqrt{v(v + 2x)} .$$

The results thus obtained show large errors especially when s is large. Therefore it seems to be worthwhile to divide the interval $(0, \infty)$ of the integral in (1.4) into subintervals $(0,t_1)$, (t_1,t_2), ..., where t_n is the n-th zero of $\cos(st)$.

Substituting

$$t = \frac{\pi}{s} (n + \frac{v}{2}), \qquad -1 \leqslant v \leqslant 1, \tag{2.4}$$

into (1.4) we get:

$$K_{is}(x) = \frac{\pi}{2s} \sum_{n=0}^{\infty} (-1)^n G_n(s,x), \tag{2.5}$$

where

$$\begin{aligned} G_0(s,x) &= \int_0^1 f(v,0)\, dv = \frac{1}{2} \int_{-1}^1 f(v,0)\, dv, \\ G_n(s,x) &= \int_{-1}^1 f(v,n)\, dv, \qquad n = 1, 2, \ldots \end{aligned} \tag{2.6}$$

and

$$f(v,n) = \cos(\frac{\pi}{2} v) \exp(-x \cosh(\frac{\pi}{2s} (2n + v))). \tag{2.7}$$

Some difficulties are expected in computing these integrals by the use of a quadrature formula of Gauss-Legendre type, i.e.

$$\int_{-1}^{1} f(v)\, dv = \sum_{i=1}^{n} a_i\, f(v_i) \; . \tag{2.8}$$

One of them is due to the slow convergence of the series (2.5), when s is large and x is small. The other is the very rapid decay of the integrand $f(v,n)$ in the interval corresponding to a specific value of n, when s is small and x is large.

These situations will be understood when we consider where the "envelope" of the integrand

$$g(z) = \exp(-x \cosh(\tfrac{\pi}{2s} z)), \qquad z = 2n + v, \tag{2.9}$$

vanishes in the sense that

$$g(z)/g(0) = 10^{-N} \; . \tag{2.10}$$

The root of this equation can be obtained easily:

$$z_c = \frac{2s}{\pi} \log_e(\xi + \sqrt{\xi^2 - 1}) \; , \qquad \xi = (N/M)/x + 1, \quad 1/M = \log_e 10. \tag{2.11}$$

Let n_c be the value of n which corresponds to the interval where z_c appears, and v_c be the value of v corresponding to z_c, then we have

$$n_c = [[z_c]/2 + 0.5], \qquad v_c = z_c - 2n_c \; . \tag{2.12}$$

Neglecting the terms G_n's preceeded by G_{nc}, we get the following formula in place of (2.5):

$$K_{is}(x) = \frac{\pi}{2s} \sum_{n=0}^{n_c} (-1)^n\, G_n(s,x) \; . \tag{2.13}$$

In some cases z_c becomes less than unity, and $K_{is}(x)$ can be computed simply by:

$$K_{is}(x) = \frac{\pi}{2s} G_0(s,x) , \qquad (2.14)$$

(see Fig.1).

In such a case, considerable errors will result if we apply, for instance, a (2m+1)-point formula of the type (2.8) directly to $G_0(s,x)$, i.e.

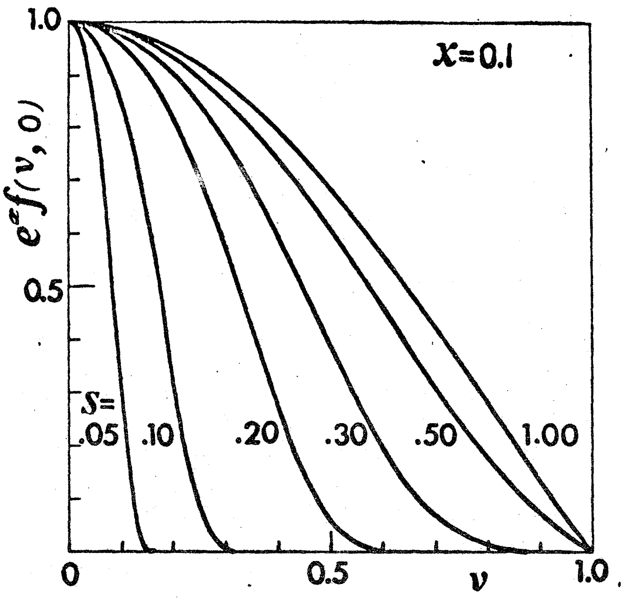

Fig.1. The function f(v,0)

$$\begin{aligned} G_0(s,x) &\\ &= \frac{1}{2}\int_{-1}^{1} f(v,0)\, dv \\ &= \frac{1}{2} a_0 \; f(0,0) + \sum_{i=1}^{m} a_i \; f(v_i,0) , \end{aligned}$$

because the points v_i's which exist to the right of the point v_c do not contribute to the integral at all.

This difficulty can be avoided by a simple modification, such as

$$\begin{aligned} G_0(s,x) &= \int_0^1 f(v,0)\, dv = \int_0^{v_c} f(v,0)\, dv \\ &= \frac{1}{2} v_c \int_{-1}^{1} f(v_c w,0)\, dw, \end{aligned}$$

or

$$G_0(s,x) = v_c\left(\frac{1}{2} a_0 \, f(0,0) + \sum_{i=1}^{m} a_i \; f(v_c z_i,0)\right), \qquad (2.15)$$

where a_i and z_i $(i=0,1,\ldots,m)$ are the weights and pivots in the (2m+1)-point quadrature formula of Gauss-Legendre type.

Since the similar situation will occur at the interval corresponding to n_c also in the case where $n_c \neq 0$, we should use the following formula to compute G_{nc}:

$$G_{nc}(s,x) = \int_{-1}^{v_c} f(v, n_c)\, dv = c \int_{-1}^{1} f(d+cw, n_c)\, dw, \tag{2.16}$$

or

$$G_{nc}(s,x) = c\,(a_0 f(d, n_c) + \sum_{i=1}^{m} a_i (f(d-cz_i, n_c) + f(d+cz_i, n_c)), \tag{2.17}$$

where $c = (v_c + 1)/2$ and $d = (v_c - 1)/2$.

All G_n's other than G_{nc} can be computed in the normal way, i.e.

$$G_0(s,x) = \frac{1}{2} a_0 e^{-x} + \sum_{i=1}^{m} a_i f(z_i, 0), \tag{2.18}$$

$$G_n(s,x) = a_0 f(0,n) + \sum_{i=1}^{m} a_i (f(-z_i,n) + f(z_i,n)). \tag{2.19}$$

As to the difficulty due to the slow convergence of the series (2.5) when x is small and s is large, the number of terms in (2.13), i.e. n_c+1 is essential. The following example will suggest the amount of computation necessary to get a value of $K_{is}(x)$ for a given pair of s and x:

	$x=0.10$	0.05	0.01
$s=\pi$,	$n_c=6$	7	8
$s=2\pi$,	12	14	17
$s=5\pi$,	31	35	42

In this connection, we have to consider the cancellation between two adjucent terms in the series of (2.13), expecially when x is small and s is large. Such a cancellation can be roughly estimated by comparing the value of the function $g(z)$ given by (2.9) at $z = 0, 1, \ldots,$ or

$$g(n) = \exp(-x \cosh(n\pi/s)).$$

Table 1. The value of g(n) for $s=5\pi$ and $x=0.01$.

n	0	1	2	3	4	5	6	7
g(n)	.9900	.9899	.9893	.9882	.9867	.9847	.9821	.9787
n	8	9	10	11	12	13	14	15
g(n)	.9746	.9694	.9631	.9554	.9460	.9208	.9042	.8793
n	16	17	18	19	20	21	22	23
g(n)	.8793	.8607	.8327	.7996	.7610	.7164	.6654	.6081
n	24	25		30	35	40		42
g(n)	.5447	.4761		.1330	.00416	.00000034		$<10^{-10}$

It can be seen from this table that considerable cancellations between G_n and G_{n+1} will occur when $n < 20$. The number of decimal places lost in the computation of the series (2.13), however, will not exceed three in case $s \leqslant 15$ and $x \geqslant 0.01$.

3. Procedures "Kisimps(s,x)" and "Ki(s,x)"

The procedure Kisimps(s,x) was written to compute exact values of $K_{is}(x)$ using an algorithm similar to that of the last example in the "Report on the Algorithmic Language ALGOL" (so-called ALGOL 58) [2]. That is, the values of $G_n(s,x)$ are computed by the use of the Simpson's one-third rule instead of the Gaussian quadrature formula. Since the double-precision arithmetic operations are used, the cancellation between G_n and G_{n+1} will have no serious effects except on the less significant digits.

The procedure Ki(s,x) is based on the method discussed in Section 2, and is aimed at computing the values of $K_{is}(x)$ to eight decimal digits in fairly large ranges of the parameters.

In this procedure, 25-point formula is used in (2.15) and (2.18), and 15-point formula in (2.19). The value N/M=24.0 is used in (2.11).

4. Numerical Results

We can see from Table 2 that the procedure Ki(s,x) gives satisfactory results except the cases where s and x are both very small. Considering that in such cases n_c is zero and v_c is very small, the errors of Ki(s,x) seem to due to those of the quadrature formula used (m=12). However, even in the case s=0.01 and x=0.01, the discrepancy between the results obtained by these two procedures appears at the ninth decimal place. Therefore, it can be concluded that the algorithm used in Ki(s,x) is capable to yield at least eight significant digits in the range $15 > s \geqslant 0.01$ and $x \geqslant 0.01$.

Table 2 Comparison of Ki(s,x) and Kisimps(s,x)

s,x	Ki(s,x)	Kisimps	n_c
s=0.01			
x=0.01	4.7191 4290	4.7191 4293 0	0
.02	4.0270 7684	4.2070 7681 3	0
.03	3.6224 7903	3.6224 7900 5	0
.05	3.1135 1503 3	3.1135 1503 2	0
0.10	2.4266 7164 6	2.4266 7164 8	0
.20	1.7525 0948 6	1.7525 0948 6	0
.30	1.3723 4171 9	1.3723 4171 8	0
.50	0.9243 6254 17	0.9243 6254 17	0
1.00	0.4210 0904 79	0.4210 0904 79	0
2.00	0.1138 9151 17	0.1138 9151 17	0
3.00	0.0347 3899 845	0.0347 3899 845	0
5.00	0.0036 9106 4499	0.0036 9106 4499	0

s=0.02			
x=0.01	4.7128 4130	4.7128 4135 1	0
.02	4.0229 3727	4.0229 3723 6	0
.03	3.6193 2872	3.6193 2869 9	0
.05	3.1113 5876 0	3.1113 5875 8	0
0.10	2.4254 7980 9	2.4254 7981 1	0
.20	1.7519 2647 9	1.7519 2647 8	0
.30	1.3719 8674 1	1.3719 8674 1	0
.50	0.9241 9297 04	0.9241 9297 04	0
1.00	0.4209 6288 00	0.4209 6288 00	0
2.00	0.1138 8442 88	0.1138 8442 88	0
3.00	0.0347 3748 070	0.0347 3748 070	0
5.00	0.0036 9096 2998	0.0036 9096 2998	0
s=0.03			
x=0.01	4.7023 5139	4.7023 5144 1	0
.02	4.0160 4455	4.0160 4452 1	0
.03	3.6140 8251	3.6140 8249 2	0
.05	3.1077 6736 8	3.1077 6736 6	0
0.10	2.4234 9438 6	2.4234 9438 8	0
.20	1.7509 5513 5	1.7509 5513 5	0
.30	1.3713 9527 6	1.3713 9527 6	0
.50	0.9239 1040 88	0.9239 1040 89	0
1.00	0.4208 8594 31	0.4208 8594 31	0
2.00	0.1138 7262 48	0.1138 7262 48	0
3.00	0.0347 3495 124	0.0347 3495 124	0
5.00	0.0036 9079 3834	0.0036 9079 3834	0
s=0.05			
x=0.01	4.6688 9036	4.6688 9040 9	0
.02	3.9440 4305	3.9940 4302 4	0
.03	3.5793 3074	3.5793 3072 0	0
.05	3.0962 9503 1	3.0962 9503 0	0
0.10	2.4171 4920 6	2.4171 4920 7	0
.20	1.7478 4965 9	1.7478 4965 9	0
.30	1.3695 0395 9	1.3695 0395 9	0
.50	0.9230 0669 36	0.9230 0669 36	0
1.00	0.4206 3982 75	0.4206 3982 75	0
2.00	0.1138 3485 97	0.1138 3485 97	0
3.00	0.0347 2685 813	0.0347 2685 813	0
5.00	0.0036 9025 2559	0.0036 9025 2559	0

Fig.2 illustrates the behavior of the function $K_{is}(x)$ in a three-dimensional form.

It should be noted that the function $K_{is}(x)$, $s > 0$,

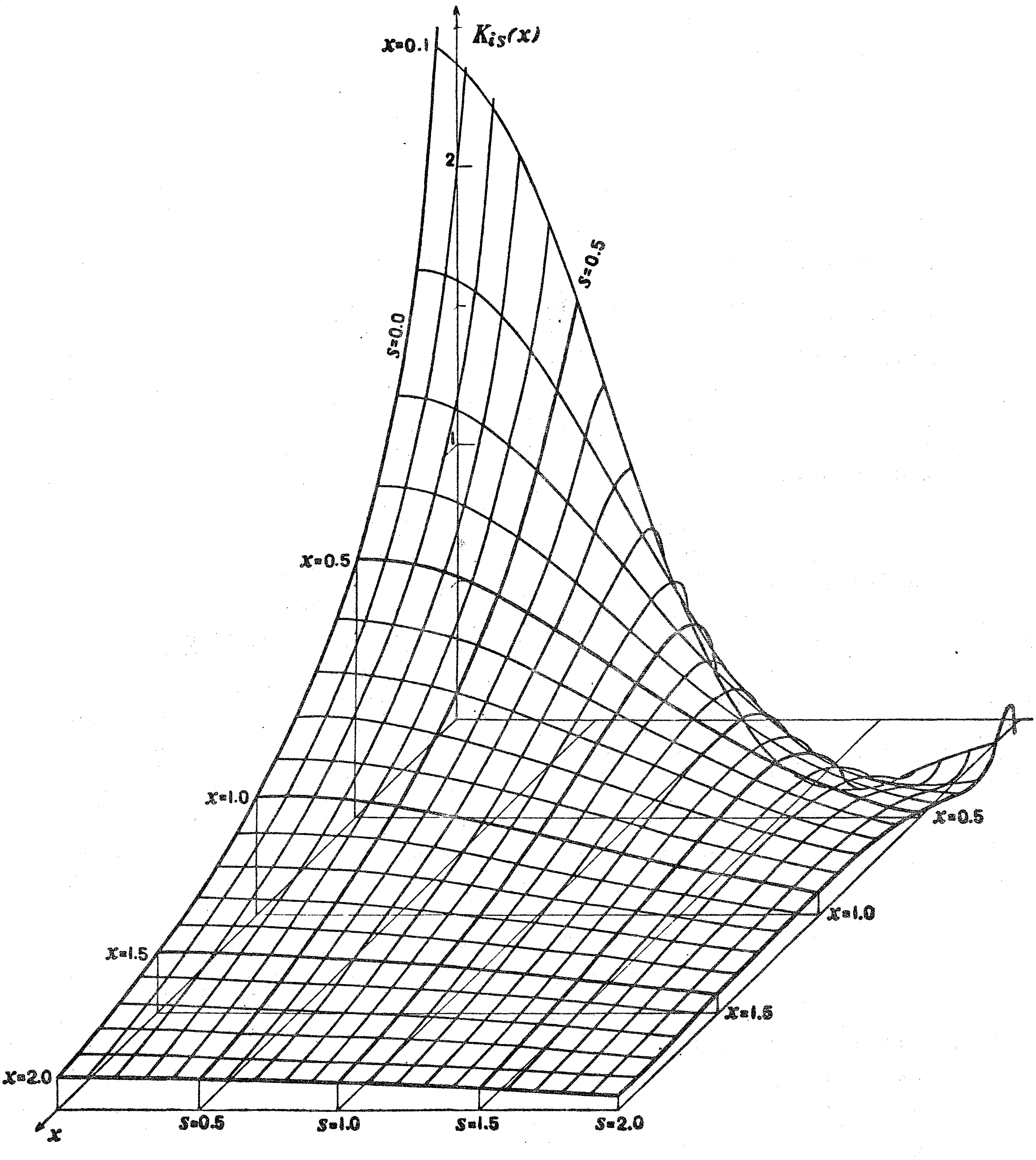

Fig. 2. $K_{is}(x)$ as function of s and x.

oscillates very rapidly, when x approaches to zero, keeping its amplitude nearly constant. (See Fig.3.)

In spite of the property of $K_0(x)$:

$$\lim_{x \to 0} K_0(x) = \infty , \qquad \lim_{x \to 0} K_0'(x) = -\infty ,$$

the values of $K_{is}(x)$, where s is positive and x is nearly zero, are finite but indeterminate, although it can be shown from (1.3) that

$$K_{is}(0) = \frac{1}{\cosh \frac{s\pi}{2}} \int_0^{\infty} \cos(st)\, dt,$$

and from (1,4) that

$$K_{is}(x) = \int_0^{\infty} \cos(st)\, dt,$$

making $K_{is}(0) = 0$.

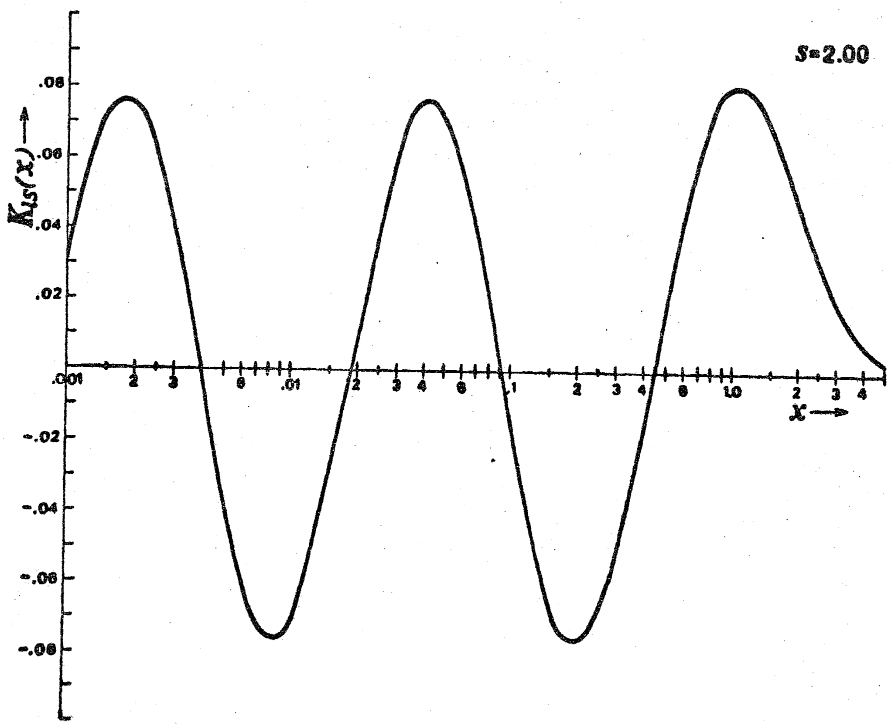

Fig.3. Oscillation of $K_{is}(x)$ near $x=0$.

4. Computing Time

The algorithm proposed here takes considerable time to get a value of $K_{is}(x)$ mainly due to the considerable number of values of the function $f(v,n)$ defined by (2.7) necessary in the quadrature. Taking into account the preparatory computaions to determine n_c and v_c for given values of s and x, the approximate total time necessary to get a value of $Ki(s,x)$ can be estimated by:

$$T = (t_n + t_q + 2t_r) + (30n_c + 25)t_x + (15n_c + 12)t_c ,$$

where t_n, t_q, t_r, t_x and t_c are computing time for $\ln(x)$, sqrt (x), entier(x), exp(x) and cos(x), respectively.

If we assume a case where

$$t_q=100, \quad t_c=t_x=250, \quad t_n=200, \text{ and } t_r=50 \quad \text{microsec.},$$

then T becomes

$$T = 9650 + 11250*n_c \quad \text{microsec.}$$

As can be seen from this formula, the time T depends strongly on the values of n_c given by (2.12).

5. Conclusion

The procedure $Ki(s,x)$ proposed here is capable to yield very exact values of the function $K_{is}(x)$ as verified by comparison with the results of a more elaborated method (Kisimps).

However, such an algorithm based on the integral representation of the function is sometimes very time-consuming.

A more efficient computing method seems to be obtained by the use of the series expansions of this function. The authors are now considering in such a direction, to establish a practical algorithm, which will be published in near future.

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References

[1] G.N.Watson: Bessel Functions, pp.181-183 (1922).

[2] A.J.Perlis and K.Samelson: Numerische Mathematik, 1, S.60 (1959).

Appendix 1. Table of the Function $K_{is}(x)$

The values of $K_{is}(x)$ for

$s = 0.01(0.01)0.05,\ 0.10(0.10)0.70,\ 1.00$ and

$x = 0.01(0.01)0.10,\ 0.20(0.10)1.00,\ 1.50(0.50)5.00$

were computed by Kisimps(s,x), and all others were computed by Ki(s,x).

x	s=0.01	s=0.02	s=0.03	s=0.04	s=0.05
0.01	4.7191 429	4.7128 414	4.7023 514	4.6876 923	4.6688 904
2	4.0270 768	4.0229 372	4.0160 445	4.0064 085	3.9940 430
3	3.6224 790	3.6193 287	3.6140 825	3.6067 468	3.5973 307
4	3.3356 885	3.3331 304	3.3288 700	3.3229 121	3.3152 631
0.05	3.1135 150	3.1113 588	3.1077 674	3.1027 444	3.0962 950
6	2.9322 584	2.9303 956	2.9272 928	2.9229 529	2.9173 800
7	2.7792 717	2.7776 339	2.7749 059	2.7710 901	2.7661 896
8	2.6470 030	2.6455 440	2.6431 137	2.6397 141	2.6353 479
9	2.5305 793	2.5292 661	2.5270 786	2.5240 184	2.5200 879
0.10	2.4266 716	2.4254 798	2.4234 944	2.4207 168	2.4171 492
20	1.7525 095	1.7519 265	1.7509 551	1.7495 960	1.7478 497
30	1.3723 417	1.3719 867	1.3713 953	1.3705 676	1.3695 040
40	1.1144 496	1.1142 112	1.1138 139	1.1132 579	1.1125 433
0.50	0.9243 6254	0.9241 9297	0.9239 1041	0.9235 1494	0.9230 0669
60	0.7774 8034	0.7773 5510	0.7771 4640	0.7768 5430	0.7764 7887
70	0.6604 8818	0.6603 9314	0.6602 3477	0.6600 1311	0.6597 2820
80	0.5653 2257	0.5652 4897	0.5651 2632	0.5649 5464	0.5647 3398
90	0.4867 1100	0.4866 5308	0.4865 5656	0.4864 2147	0.4862 4782
1.00	0.4210 0905	0.4209 6288	0.4208 8594	0.4207 7825	0.4206 3983
1.50	0.2137 9992	0.2137 8299	0.2137 5479	0.2137 1530	0.2136 6455
2.00	0.1138 9151	0.1138 8443	0.1138 7262	0.1138 5610	0.1138 3486
2.50	0.0623 4648 7	0.0623 4328 9	0.0623 3795 9	0.0623 3049 8	0.0623 2090 6
3.00	0.0347 3899 8	0.0347 3748 1	0.0347 3495 1	0.0347 3141 0	0.0347 2685 8
3.50	0.0195 9864 8	0.0195 9790 2	0.0195 9665 9	0.0195 9491 8	0.0195 9268 1
4.00	0.0111 5955 1	0.0111 5917 4	0.0111 5854 7	0.0111 5766 8	0.0111 5653 9
4.50	0.0063 9979 26	0.0063 9959 87	0.0063 9927 55	0.0063 9882 31	0.0063 9824 15
5.00	0.0036 9106 45	0.0036 9096 30	0.0036 9079 38	0.0036 9055 70	0.0036 9025 26

x	s=0.10	s=0.20	s=0.30	s=0.40	s=0.50
0.01	4.5141 924	3.9297 807	3.0698 503	2.0783 686	1.1098 861
2	3.8920 254	3.5018 801	2.9119 673	2.2012 569	1.4597 742
3	3.5195 361	3.2201 279	2.7610 211	2.1954 485	1.5859 465
4	3.2520 096	3.0075 497	2.6292 608	2.1565 009	1.6364 071
0.05	3.0429 253	2.8360 376	2.5137 600	2.1067 995	1.6524 458
6	2.8712 390	2.6919 523	2.4112 338	2.0539 088	1.6504 480
7	2.7255 990	2.5675 788	2.3191 327	2.0008 486	1.6382 143
8	2.5991 698	2.4581 048	2.2355 521	1.9489 176	1.6199 033
9	2.4875 109	2.3603 157	2.1590 575	1.8986 756	1.5979 095
0.10	2.3875 716	2.2719 528	2.0885 485	1.8503 386	1.5736 895
20	1.7333 499	1.6762 849	1.5844 273	1.4624 096	1.3162 514
30	1.3606 662	1.3257 704	1.2692 014	1.1932 480	1.1009 282
40	1.1066 033	1.0831 012	1.0448 360	0.9931 1783	0.9296 8957
0.50	0.9187 8030	0.9020 3482	0.8746 8770	0.8375 5619	0.7917 3431
60	0.7733 5628	0.7609 7115	0.7406 9923	0.7130 8025	0.6788 4015
70	0.6573 5807	0.6479 4971	0.6325 2301	0.6114 4934	0.5852 3002
80	0.5628 9803	0.5556 0530	0.5436 3058	0.5272 3742	0.5067 8268
90	0.4848 0282	0.4790 5987	0.4696 1884	0.4566 7137	0.4404 7762
1.00	0.4194 8783	0.4149 0726	0.4073 6964	0.3970 1711	0.3840 4302
1.50	0.2132 4199	0.2115 5924	0.2087 8101	0.2049 4625	0.2001 0833
2.00	0.1136 5799	0.1129 5299	0.1117 8684	0.1101 7262	0.1081 2833
2.50	0.0622 4102 7	0.0619 2245 2	0.0613 9483 0	0.0606 6311 3	0.0597 3413 3
3.00	0.0346 8894 5	0.0345 3767 7	0.0342 8692 7	0.0339 3871 9	0.0334 9585 3
3.50	0.0195 7404 2	0.0194 9965 4	0.0193 7626 2	0.0192 0474 2	0.0189 8630 5
4.00	0.0111 4713 2	0.0111 0958 1	0.0110 4726 1	0.0109 6056 6	0.0108 5004 2
4.50	0.0063 9339 67	0.0063 7405 22	0.0063 4193 45	0.0062 9722 71	0.0062 4018 47
5.00	0.0036 8771 63	0.0036 7758 80	0.0036 6076 63	0.0036 3733 88	0.0036 0742 71

x	s=0.60	s=0.70	s=0.80	s=0.90	s=1.00
0.01	0.2974 7093	-0.2720 4894	-0.5700 6828	-0.6236 5445	-0.5006 3372
2	0.7733 3270	+0.2100 2906	-0.1889 6242	-0.4128 2424	-0.4786 0842
3	0.9950 4998	0.4765 3852	+0.0687 0154 5	-0.2093 6784	-0.3580 6366
4	1.1175 429	0.6436 9282	0.2487 5464	-0.0465 5127 9	-0.2357 8658
0.05	1.1899 357	0.7558 0181	0.3798 7162	+0.0824 4481 4	-0.1270 3351
6	1.2334 296	0.8339 1897	0.4782 5145	0.1856 2172	-0.0332 5508 4
7	1.2588 290	0.8895 0528	0.5536 2884	0.2690 9121	+0.0470 1706 6
8	1.2722 558	0.9293 9606	0.6122 1646	0.3372 9649	0.1157 2325
9	1.2774 453	0.9579 4421	0.6581 7846	0.3934 8134	0.1746 6626
0.10	1.2768 049	0.9780 6227	0.6944 1819	0.4400 5228	0.2253 8189
20	1.1529 420	0.9799 6795	0.8048 2530	0.6345 5426	0.4753 3346
30	0.9958 1750	0.8818 5149	0.7631 1347	0.6436 2299	0.5271 3838
40	0.8566 4235	0.7763 1581	0.6911 8939	0.6037 7069	0.5164 8739
0.50	0.7385 4609	0.6794 8980	0.6161 7620	0.5502 6406	0.4833 9609
60	0.6388 6318	0.5941 5839	0.5458 2210	0.4949 9824	0.4428 3818
70	0.5544 7882	0.5199 0069	0.4822 6787	0.4423 9426	0.4011 0918
80	0.4827 0506	0.4555 1106	0.4257 5920	0.3940 4285	0.3609 7256
90	0.4213 5847	0.3996 8603	0.3758 7284	0.3503 6006	0.3236 0524
1.00	0.3686 8651	0.3512 2596	0.3319 7136	0.3112 5605	0.2894 2804
1.50	0.1943 3388	0.1877 0142	0.1802 9972	0.1722 2602	0.1635 8399
2.00	0.1056 7660	0.1028 4427	0.0996 6193 4	0.0961 6347 6	0.0923 8546 0
2.50	0.0586 1650 2	0.0573 2048 8	0.0558 5787 0	0.0542 4176 8	0.0524 8646 1
3.00	0.0329 6186 7	0.0323 4099 5	0.0316 3811 2	0.0308 5867 6	0.0300 0865 9
3.50	0.0187 2248 6	0.0184 1512 7	0.0180 6635 3	0.0176 7855 5	0.0172 5435 7
4.00	0.0107 1638 4	0.0105 6042 4	0.0103 8312 8	0.0101 8558 3	0.0099 6898 73
4.50	0.0061 7113 06	0.0060 9045 41	0.0059 9860 69	0.0058 9609 89	0.0057 8349 40
5.00	0.0035 7118 61	0.0035 2880 22	0.0034 8049 20	0.0034 2650 07	0.0033 6710 00

x	s=1.10	s=1.20	s=1.30	s=1.40	s=1.50
0.01	-0.2874 8367	-0.0664 9504 4	+0.1019 5904	+0.1883 7470	0.1935 2416
2	-0.4239 3795	-0.2969 1556	-0.1456 2949	-0.0094 4330 86	+0.0864 1546 8
3	-0.3944 1680	-0.3469 3317	-0.2492 9551	-0.1341 8611	-0.0283 1297 8
4	-0.3257 3428	-0.3335 5401	-0.2827 9793	-0.1990 5719	-0.1059 8424
0.05	-0.2497 3831	-0.2958 4943	-0.2819 3332	-0.2278 1816	-0.1534 8467
6	-0.1763 1702	-0.2492 4473	-0.2636 9719	-0.2350 2632	-0.1798 4510
7	-0.1086 1496	-0.2004 8679	-0.2367 3456	-0.2292 5110	-0.1917 7113
8	-0.0473 8800 4	-0.1526 7368	-0.2056 9269	-0.2156 5012	-0.1938 7787
9	+0.0075 0436 27	-0.1072 2447	-0.1731 6417	-0.1974 1419	-0.1893 1505
0.10	0.0565 2881 5	-0.0647 3886 1	-0.1406 2454	-0.1765 6411	-0.1802 4888
20	0.3321 6211	+0.2086 4979	+0.1069 2333	+0.0276 4915 2	-0.0298 4021 0
30	0.4169 8546	0.3159 2213	0.2260 4555	0.1487 4507	+0.0847 0067 2
40	0.4315 8848	0.3510 5986	0.2765 5835	0.2093 6650	0.1503 6960
0.50	0.4171 3854	0.3529 2714	0.2920 2183	0.2354 7202	0.1840 9341
60	0.3904 6201	0.3389 2280	0.2891 7546	0.2420 5108	0.1982 3785
70	0.3592 3167	0.3175 4617	0.2767 8069	0.2375 8810	0.2005 3129
80	0.3271 5839	0.2931 9305	0.2596 3633	0.2270 0141	0.1957 4341
90	0.2960 6990	0.2682 0760	0.2404 5263	0.2132 0992	0.1868 4630
1.00	0.2668 4112	0.2438 4624	0.2207 8323	0.1979 7325	0.1757 1212
1.50	0.1544 8179	0.1450 2991	0.1353 3916	0.1255 1867	0.1156 7397
2.00	0.0883 6656 4	0.0841 4696 1	0.0797 6769 5	0.0752 7006 5	0.0706 9501 7
2.50	0.0506 0719 1	0.0486 1995 5	0.0465 4129 3	0.0443 8807 2	0.0421 7727 1
3.00	0.0290 9447 8	0.0281 2291 7	0.0271 0104 9	0.0260 3615 4	0.0249 3563 7
3.50	0.0167 9659 5	0.0163 0828 1	0.0157 9257 7	0.0152 5275 9	0.0146 9218 4
4.00	0.0097 3464 03	0.0094 8392 88	0.0092 1831 43	0.0089 3931 93	0.0086 4851 42
4.50	0.0056 6140 49	0.0055 3048 82	0.0053 9143 81	0.0052 4498 11	0.0050 9186 99
5.00	0.0033 0258 57	0.0032 3327 58	0.0031 5950 76	0.0030 8163 51	0.0030 0002 65

DISCUSSION

Prof. H. Takahashi suggested that the expression (1.4) is suited for the trapezoidal rule rather than the Gauss-Legendre formula, and the amount of computation will be remarkably reduced by the use of the former.

In this connection, one of the authors calculated the value of $K_{is}(x)$ in the case of $s=0.05$ and $x=0.10$, using different formulas, and obtained the following results:

Calculated values of $K_{is}(x)$ for $s=0.05$ and $x=0.10$;

$n_c = 0$, $z_c \fallingdotseq 0.20$.

Trapezoidal rule:

step*	number of pivots	$K_{is}(x)$	theoretical error@
0.05	4	2.4216 984	$10^{-2.7}$
0.04	5	2.4170 804	$10^{-3.4}$
0.03	7	2.4172 028	$10^{-4.5}$
0.02	10	2.4171 4915	$10^{-6.8}$
0.01	20	2.4171 4920 7	$10^{-13.6}$

Legendre-Gauss (2m+1)-point formula:

m	number of pivots	$K_{is}(x)$
7**	15	2.4171 4919
7***	8	2.4171 648
12***	13	2.4171 4920 6
	(exact)	2.4171 4920 7

@ estimated by Takahashi-Mori's method: $\exp(-\pi^2/\Delta t)$.

* $\Delta v = (2s/\pi)\Delta t$, ($t$ in (1.4)).
** applied to the interval (0,1).
*** applied to the interval (-1,1).

These results show that both the trapezoidal rule and Gauss-Legendre formula yield nearly equal accuracy, if the same number of pivots is used, and no considerable reduction of computing time can be expected in such a case.

On the other hand, in case the value of x is large, the situation will become different. According to the Takahashi-Mori's method, the error of the trapezoidal quadrature applied to (1.4) will be given by:

$$E \fallingdotseq e^{-\pi^2/\Delta t} = e^{-2\pi s/\Delta v}, \tag{i}$$

if the condition

$$s + \pi/\Delta t > x \quad \text{or} \quad 1 + (2/\Delta v) > x/s \tag{ii}$$

holds. Combining (i) and (2.11) we can estimate the number of pivots necessary to keep $E \leq 10^{-N}$:

$$p = z_c/\Delta v \simeq 0.537N \log_{10}(4.605N/x + 2). \tag{iii}$$

It is remarkable that the number p does not depend on the parameter s. For $N=10$, we obtain:

x=0.005	0.01	0.02	0.05	0.10	0.20	0.50	1.00	2.00
p= 21.3	19.7	18.1	15.9	14.3	12.7	10.6	9.0	7.5

These results show that the trapezoidal rule will become more and more advantageous as x increases, in accordance whith the suggestion given by Prof. Takahashi.

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